On (r, c)-constant circulants

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Introducing (r, c)-constant graphs

Definition

A graph G is an (r, c)-constant graph if, and only if, every vertex v has degree r and exactly c edges in the open-neighbourhood.

- This is a natural extension of *graphs with constant-link*, which are those r-regular graphs such that the subgraphs induced by the open-neighbourhoods of any two vertices are isomorphic. Therefore every open-neighbourhood has a constant number c of edges.
- Similarly, (r, c)-constant graphs extend the notion of (r, b)-regular graphs, which are those r-regular graphs such that every open-neighbourhood is induces a b-regular subgraph. Therefore, once again, every open-neighbourhood has a constant number c of edges.
- **Notation:** By $e(v)$ we will denote the number of edges in the subgraph induced by the open-neighbourhood of a vertex v.

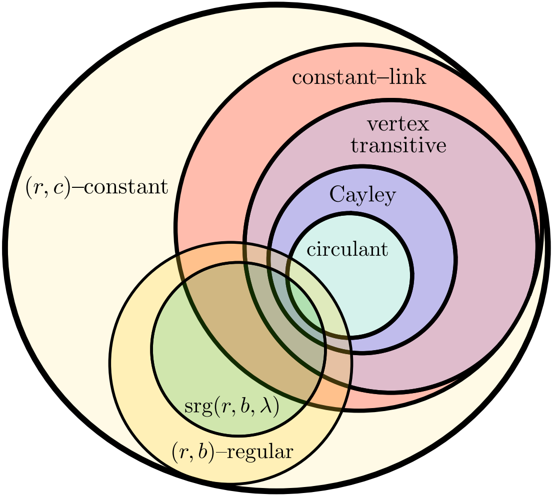

Figure: Interesting sub-families of graphs within the family of (r, c)-constant graphs.

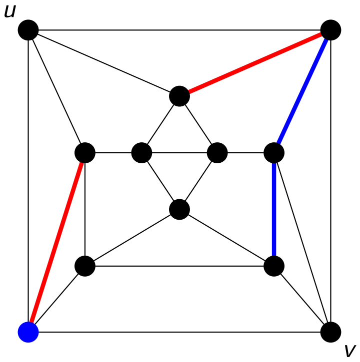

Figure: An example of a $(4, 2)$-constant graph, which is also a planar graph.

- The subgraph induced by the open-neighbourhood of u is $2K_2$ (highlighted in red), whilst the subgraph induced by the open-neighbourhood of v is $K_1 \cup P_3$ (highlighted in blue), and therefore they are not isomorphic.
- Also, in the case of v the open-neighbourhood does not induce a regular graph.
- Hence this is an example of a (planar) (r, c)-constant graph that is neither a graph with constant-link nor an (r, b)-regular graph.

- The family of (r, c)-constant graphs was recently introduced to demonstrate the existence of k-flip graphs, which Josef Lauri talks about in his presentation.
- A number of results on (r, c)-constant graphs have been established, such as that for $r \geq 1$ and $0 \leq c \leq \frac{r^2}{2} - 5r^{\frac{3}{2}}$, there exists an (r, c)-graph. The proof is constructive and follows from the feasibility of line graphs problem, which Christina Zarb talks about in her presentation.
- Together with Yair Caro, the existence problem of planar and circulant (r, c)-constant graphs was studied, as well as the problem of establishing the smallest order of an (r, c)-constant graph for given values r and c.

Today we will focus on the existence and construction of (r, c)-circulants...

Existence and construction of (r, c)-circulants

- Consider the circulant $\mathrm{Circ}\,(n, S)$ where n is the number of vertices and S is the set of *jumps* $i \in S$ where $1 \leq i \leq \frac{n}{2}$. **Note:** $\mathrm{Circ}\,(n, S) = \mathrm{Cay}\left(\mathbb{Z}_n; S \cup S^{-1}\right)$.
- We are interested in the existence problem of (r, c)-constant graphs which are circulants (termed as (r, c)-circulants).
- The motivation for this problem is two-fold: (r, c)-circulants are useful for constructing small 2-flip graphs, and exhaustive computer searches yielded that in many cases the smallest order of an (r, c)-graph is realised by a circulant.

Problem

Given a non-negative integer c and a positive integer r, does an (r, c)-circulant graph exist?

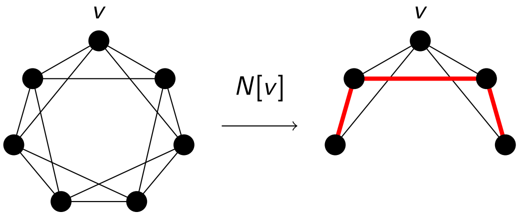

Figure: The $(4, 3)$-circulant $\text{Circ}(7, \{1, 2\})$ where $c \equiv 0 \pmod 3$

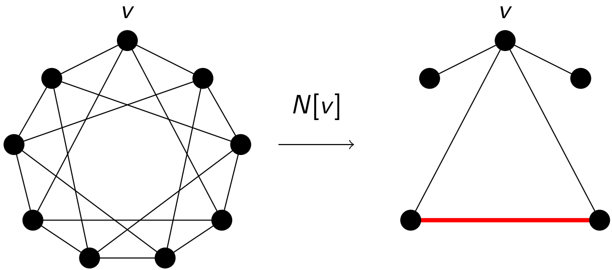

Figure: The $(4, 1)$-circulant $\text{Circ}(9, \{1, 3\})$ where $c \equiv 1 \pmod 3$

We briefly illustrate the ideas behind constructing (r, c)-circulants for a given c.

Let k, j be integers such that $k > j \geq 0$. Consider:

- $S_{k,j} = \{1, \ldots, k-1, k+j\}$ and $\mathrm{Circ}\left(n, S_{k,j}\right)$ which is $2k$-regular.
- For every vertex v:

$$e(v) = \begin{cases} 3\binom{k-1}{2} + 3(k-1-j) + 1, & n = 3(k-1) \\ 3\binom{k-1}{2} + 3(k-1-j), & n \text{ sufficiently larger than } 3(k-1) \end{cases}$$

- Observe that for appropriate choices of n, k and j we can construct circulants with $e(v) = c$ for all $c = 0, 1 \pmod 3$.

We can increase the degree by $2l$, simply by adding to $S_{k,j}$ the first l terms of some arithmetic progression with difference $d = k + j + 1$ and a suitably large n. Let $R_{k,j,l}$ be the set containing these l terms.

- Observe that the choice d is such that in $\mathrm{Circ}\left(n, S_{k,j} \cup R_{k,j,l}\right)$ the number of edges $e(v)$ is the same as in $\mathrm{Circ}\left(n, S_{k,j}\right)$. In other words, we are only increasing r (controlled by l) for the c associated with k and j.

- As a consequence of this construction (with some more involved arguments), we arrive at the following theorem:

Theorem

Let c, r be a positive integers such that $c \equiv 0, 1 \pmod 3$ and $r \geq 6 + \sqrt{\frac{8c-5}{3}}$. Then an (r, c)-circulant graph exists.

- A similar construction can also be used to construct $(r, 0)$-circulants for $r \geq 1$.

We have some answers to our problem, **except for the case when** $c \equiv 2 \pmod 3$!

In fact, we have seen examples and constructions of (r, c)-circulants in the case when $c \equiv 0, 1 \pmod 3$, but none for the case when $c \equiv 2 \pmod 3$.

Here lies a surprising result about circulants...

- We first note that following result on (r,c)-constant graphs (not necessarily limited to circulants), which establishes that the case $c \equiv 1,2 \pmod 3$ may possibly arise only when $n \equiv 0 \pmod 3$:

Lemma

If G is an (r,c)-constant graph on $n \equiv 1,2 \pmod 3$ vertices, then $c \equiv 0 \pmod 3$.

Proof.

- Let $t(G)$ denote the number of triangles in G.
- Every vertex v has $e(v) = c$ and hence v is a vertex on exactly c triangles.
- Counting (with multiplicities) over all vertices, we get cn triangles.
- On the other hand, every triangle is counted this way three times, hence this double counting gives $t(G) = \frac{cn}{3}$.
- But as $n \equiv 1,2 \pmod 3$ it follows that $c \equiv 0 \pmod 3$.

□

- As a consequence of the previous lemma, an (r, c)-circulant on $n \equiv 1, 2 \pmod 3$ vertices must satisfy $c \equiv 0 \pmod 3$. Therefore all that remains is to consider circulants on $n \equiv 0 \pmod 3$ vertices.
- It turns out that for a circulant G on n vertices, if $n \equiv 0 \pmod 3$ then $e(v) \equiv 0, 1 \pmod 3$. In particular:

Lemma

For any vertex v in $\mathrm{Circ}(n, S)$, $e(v) \equiv 0 \pmod 3$ except when $n \equiv 0 \pmod 3$ and $\frac{n}{3} \in S$, in which case $e(v) \equiv 1 \pmod 3$.

This settles the existence problem for (r, c)-circulants when $c \equiv 2 \pmod 3$!

Theorem

No (r, c)-circulant exists for $c \equiv 2 \pmod 3$.

The (r, c)-graph database

- We anticipate that (r, c)-constant graphs may become useful in a number of areas, and therefore we have made publicly available an exhaustive collection of (r, c)-constant graphs, along with tools for `Mathematica` and McKay's geng for checking and generating such graphs.
- Presently, the database contains 1794 distinct (r, c)-graphs for $c > 0$, with r ranging from 2 to 776. Additionally, there are 1887 non-bipartite $(r, 0)$-graphs, as well as 1007 planar (r, c)-graphs (51 of which have $c > 0$). In the case that a graph has a constant-link, we also give what it is.

Database

https://rcgraphs.research.um.edu.mt

Thanks for attending!