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Nullity Relations on Graphs with Independent Core Vertices

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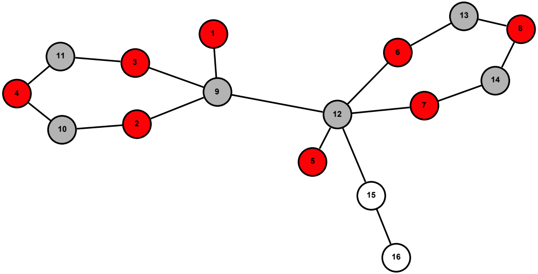
Overview

Graphs, their Adjacency Matrix, and Core Vertices



$$\left\{ \begin{pmatrix} 2 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 2 \\ -1 \\ -1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \right\}$$

$$\eta(G) = 2$$

Core Vertices

The *core vertices* (red) of a graph correspond to non-zero entries in the kernel-eigenvectors. Their set is denoted by CV.

Core-Forbidden Vertices

The *core-forbidden vertices* (grey and white) to the zero-entries in the kernel-eigenvectors.

1. neighbours of a core vertex (grey) are denoted by $N(CV)$
2. remainder (white) are denoted by CFV_{dist}

These define a partitioning of the vertex set,

$$V(G) = CV \mathbin{\dot\cup} N(CV) \mathbin{\dot\cup} CFV_{\text{dist}}$$

Adjacency Matrix Partitioning

A graph with *independent* core vertices has a $|CV| \times |CV|$ block matrix $\mathbf{0}_{CV}$ of zeros. The vertex partitioning results in a partitioning of the adjacency matrix,

$$\mathbf{A} = \left[\begin{array}{c|c|c} \mathbf{0}_{CV} & \mathbf{Q} & \mathbf{0} \\ \hline \mathbf{Q}^{\intercal} & \multicolumn{2}{c}{\multirow{2}{*}{\mathbf{B}}} \\ \cline{1-1} \mathbf{0} & \multicolumn{2}{c}{} \end{array}\right]$$

where $\mathbf{Q}$ is $CV \times N(CV)$.

Main Results

Main Result 1

Let G be a graph with independent core vertices. Then $\eta(G) = |CV| - \text{rank}(\mathbf{Q})$.

Main Result 2

Let G be a graph with independent core vertices. $\mathbf{Q}$ has linearly independent columns if and only if $\eta(G) = |CV| - |N(CV)|$.

Main Result 3

Let G be a bipartite graph with independent core vertices. If G does not contain any C_{4k} cycle, then the columns of $\mathbf{Q}$ are linearly independent if and only if $\mu = |N(CV)| + \dfrac{|Inv|}{2}$.

A Dimensional Result

Proposition 1.1

Let $\left[\begin{array}{c}\mathbf{B}_{k_1 \times h_1} \\ \hline \mathbf{C}_{k_2 \times h_1}\end{array}\right]$ be an $(k_1 + k_2) \times h$ submatrix of the $(k_1 + k_2) \times (h_1 + h_2)$ matrix $\mathbf{M}$. Then,

$$h_1 = \operatorname{rank}\left([\mathbf{B}^\intercal | \mathbf{C}^\intercal]\right) + \eta\left(\left[\begin{array}{c}\mathbf{B} \\ \hline \mathbf{C}\end{array}\right]\right)$$

Nullity Relations with Q

Lemma 2.1

Let G be a graph with independent core vertices. Then $\eta\left(\mathbf{Q}^\intercal\right) = \eta\left(\mathbf{A}\right) = \eta(G)$.

Theorem 2.2

Let G be a graph with independent core vertices. Then $\eta(G) = |CV| - \text{rank}(\mathbf{Q})$.

Proof.

The dimensional result immediately gives,

$$|CV| = \text{rank}\left([\mathbf{0}|\mathbf{Q}|\mathbf{0}]\right) + \eta\left([\mathbf{0}|\mathbf{Q}|\mathbf{0}]^\intercal\right) = \text{rank}\left(\mathbf{Q}\right) + \eta\left(\mathbf{Q}^\intercal\right)$$

Rearranging and applying Lemma 2.1, $\eta(G) = |CV| - \text{rank}(\mathbf{Q})$. □

Lemma 2.3

The columns of $\mathbf{Q}^\intercal$ are linearly dependent and $\text{rank}(\mathbf{Q}) < |CV|$.

Proposition 2.4

Let G be a graph with independent core vertices. If $|CV| \leq |N(CV)|$, then the columns of $\mathbf{Q}$ are linearly dependent.

Proof.

By $\text{rank}(\mathbf{Q}) < |CV|$ and the given, $\text{rank}(\mathbf{Q}) < |CV| \leq |N(CV)|$, and thus linear dependence follows. □

Lemma 2.5

Let G be a graph with independent core vertices. Then $\eta(G) > 0$.

Theorem 2.6

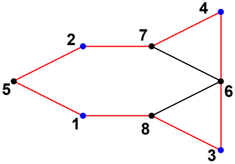
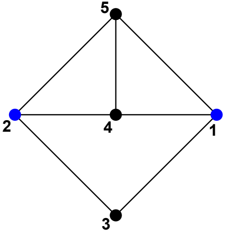
Let G be a graph with independent core vertices. **Q** has linearly independent columns if and only if $\eta(G) = |CV| - |N(CV)|$.

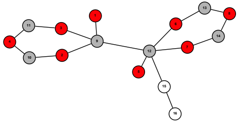
Proof.

Let **Q** have linearly independent columns. By the contra-positive of Proposition 2.4, linear independence gives $|N(CV)| < |CV|$. Because **Q** has full rank, $\text{rank}(\mathbf{Q}) = |N(CV)|$. Theorem 2.2 gives $\eta(G) = |CV| - \text{rank}(\mathbf{Q})$, hence $\eta(G) = |CV| - |N(CV)|$.

Conversely, let $\eta(G) = |CV| - |N(CV)|$. Since $\eta(G) > 0$, $|N(CV)| < |CV|$. Also by $\eta(G) = |CV| - \text{rank}(\mathbf{Q})$, $\text{rank}(\mathbf{Q}) = |N(CV)|$. Then **Q** has full column rank and thus linear independence follows. □

Characterisation on Structure of Q

Q w/ linearly dependent columns				
$\|CV\|, \|N(CV)\|$	Example	$\eta(G)$	$\|CV\|$	rk(**Q**)
$\|CV\| = \|N(CV)\|$		1	4	3
$\|CV\| < \|N(CV)\|$		1	2	1

Q w/ linearly independent columns				
$\lvert CV\rvert, \lvert N(CV)\rvert$	Example	$\eta(G)$	$\lvert CV\rvert$	rk($\mathbf{Q}$)
$\lvert N(CV)\rvert < \lvert CV\rvert$		2	8	6

$$\mathbf{Q} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

A Special Case: C_{4k}-free Bipartite Graphs

Lemma 3.1 [Gutman & Borovićanin, 2011]

Let G be a bipartite graph on n vertices. If G does not contain any C_{4k} cycle, then $\eta(G) = n - 2\mu$, where μ is the size of its maximal matching.

Theorem 3.2

Let G be a bipartite graph with independent core vertices. If G does not contain any C_{4k} cycle, then the columns of $\mathbf{Q}$ are linearly independent if and only if $\mu = |N(CV)| + \frac{|Inv|}{2}$.

Proof.

Firstly, we note that $n = |CV| + |N(CV)| + |Inv|$. Let $\mathbf{Q}$ have linearly independent columns. Thus $\eta(G) = |CV| - |N(CV)|$ and by Lemma 3.1 it follows that,

$$|CV| - |N(CV)| = |CV| + |N(CV)| + |Inv| - 2\mu$$

Rearranging gives us $\mu = |N(CV)| + \frac{|Inv|}{2}$, as desired.

Conversely, let $\mu = |N(CV)| + \frac{|Inv|}{2}$. Once again by Lemma 3.1, $\eta(G) = |CV| - |N(CV)|$. Theorem 2.6 gives us that linear independence of the columns of $\mathbf{Q}$, concluding the proof. □

Question 3.3

When does a bipartite graph have independent core vertices?