# Local v. Global Majority: An Edge-Colouring Approach

Student: Xandru Mifsud
Supervisors: Josef Lauri    Christina Zarb    Yair Caro

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L-Università ta' Malta
Faculty of Science
Department
of Mathematics

# Local v. global phenomena in graphs

- Many variations on local-global phenomena in graphs have been studied over the past 80 years, where typically some global graph parameter is studied in terms of a local parameter (eg. on some smaller induced subgraph).
- Eulerian graphs are a classical example of local v. global phenomena in graphs.

## Eulerian graphs

- A graph is said to be *Eulerian* if every component has a trail starting and ending at the same vertex, such that each edge is visited exactly once.
- (Folklore) *A graph is Eulerian if, and only if, every vertex has even degree.*
- Eulerianicity is a property of a graph and therefore is a **global property**.
- On the other hand, having even degree is a vertex-specific property and therefore is a **local property**.

# Flip colourings: a new local v. global majority problem

- We introduce a new problem on local v. global majority in graphs, concerning edge-colourings.
- In particular, we ask for which positive integers $b$ and $r$, such that $b < r$, does a $b + r$ regular graph $G$ exist with an edge colouring $f : E(G) \to \{\text{blue, red}\}$ satisfying the following:
  1. The subgraphs induced by the blue and red edges are $b$ and $r$ regular respectively, resulting in a **global majority ordering** since $b < r$, where across the entire graph 'red' wins against 'blue'.
  2. On the other-hand, for every vertex $v$, the number of blue edges in the closed neighbourhood of $v$ is *greater* than the number of red edges, resulting in a **locally opposite majority ordering** where locally 'blue' wins against 'red'.
- We term such a graph as a $(b, r)$-flip graph, due to the local v. global majority flip they demonstrate.

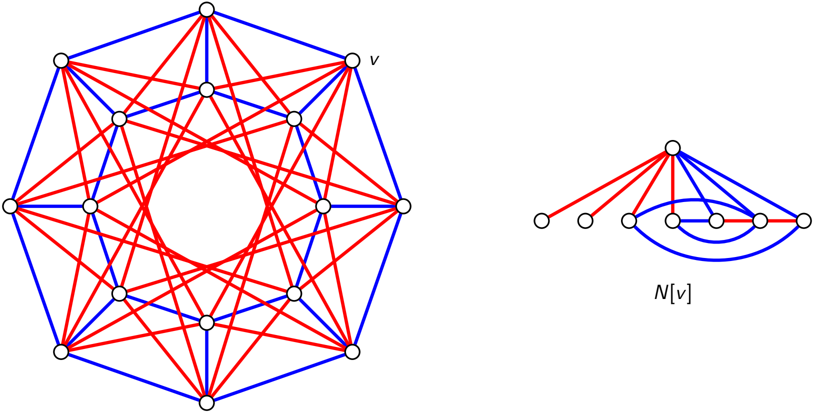


Figure: A (3, 4)-flip graph: $\deg_{\text{blue}}(v) = 3 < 4 = \deg_{\text{red}}(v)$ but $e_{\text{blue}}[v] = 7 > 6 = e_{\text{red}}(v)$.

# Extending to $k \geq 2$ colours

- Let $k \in \mathbb{N}$ and let $f\colon E(G) \to \{1, \ldots, k\}$ be an edge-colouring of $G$. Let $v$ be a vertex in $V(G)$ and $j \in \{1, \ldots, k\}$. Then:
  1. let $e_j[v]$ denote the number of edges coloured $j$ in $N[v]$,
  2. and let $\deg_j(v)$ denote the number of edges coloured $j$ incident to $v$.
- Given $k \geq 2$, a $d$-regular graph $G$ and an increasing positive-integer sequence $(a_1, \ldots, a_k)$ such that $d = \sum_{j=1}^{k} a_j$, does there exists an edge-colouring on $k$ colours such that:
  1. The edges coloured $j$ span an $a_j$-regular subgraph, namely $\deg_j(v) = a_j$ for every $v \in V$,
  2. and for every vertex $v \in V$, $e_k[v] < e_{k-1}[v] < \ldots < e_1[v]$.
- Observe how the sequence $(a_1, \ldots, a_k)$ is increasing whilst $(e_1[v], \ldots, e_k[v])$ is decreasing for every vertex $v$, *i.e.* there is a *flip*.
- If such an edge-colouring exists then $G$ is said to be an $(a_1, \ldots, a_k)$-flip graph, or more simply a $k$-flip graph, and $(a_1, \ldots, a_k)$ is called a $k$-flip sequence.

# Coloured Cartesian products of graphs

## Definition (Cartesian product)

The Cartesian product $G \square H$ of the graphs $G$ and $H$ is the graph such that $V(G \square H) = V(G) \times V(H)$ and there is an edge $\{(u, v), (u', v')\}$ in $G \square H$ if, and only if, either $u = u'$ and $v \sim v'$ in $H$, or $v = v'$ and $u \sim u'$ in $G$.

- We extend the edge-colourings of $G$ and $H$ to an edge-colouring of $G \square H$ as follows. Consider the edge $e = \{(u, v), (u', v')\}$ in $G \square H$; if $u = u'$ then $e$ inherits the colouring of the edge $\{v, v'\}$ in $H$, otherwise if $v = v'$ the colouring of the edge $\{u, u'\}$ in $G$ is inherited.
- We refer to $G \square H$ with its inherited edge-colouring as the *coloured Cartesian product* (CCP) of the graphs $G$ and $H$.
- The CCP and its properties are a highly useful tool in the construction of flip-graphs, as we will illustrate soon.

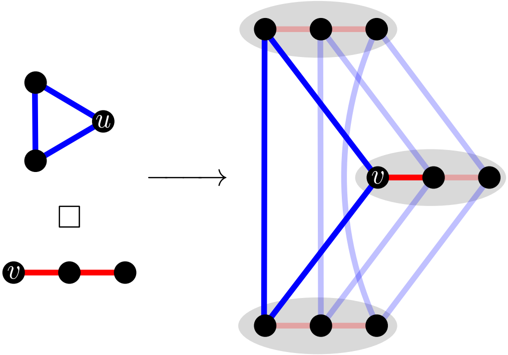


Figure: Illustration of edge-colouring inheritance in the Cartesian product graphs.

## Lemma (CCP properties)

*Let $G$ and $H$ be edge-coloured from $\{1, \ldots, k\}$. Then in the coloured Cartesian product $G \square H$, for any $1 \leq j \leq k$ and $(u, v) \in V(G \square H)$,*

1. $\deg_j\left((u, v)\right) = \deg_j^G(u) + \deg_j^H(v)$
2. $e_j\left[(u, v)\right] = e_j^G[u] + e_j^H[v]$

# Constructing 2-flip graphs

- Using the CCP and its properties, we can easily construct 2-flip graphs.
- Let $b, r \in \mathbb{N}$ and consider an $r$-regular complete bipartite graph $K_{r,r}$ with edges coloured red, and a $b$-regular complete graph $K_{b+1}$ with edges coloured blue.
- In $K_{r,r}$ every closed neighbourhood has $r$ red edges, whilst in $K_{b+1}$ every closed neighbourhood has $\binom{b+1}{2}$ blue edges.
- Consider the CCP $K_{r,r} \square K_{b+1}$. By the CCP properties every vertex has $r$ incident red edges and $b$ incident blue edges, and therefore for a flip graph we require $b < r$ (more incident red edges than blue edges).
- Moreover, every closed neighbourhood has $r$ red edges and $\binom{b+1}{2}$ blue edges, and therefore for the graph to be a flip-graph we require $r < \binom{b+1}{2}$ (more blue edges in the closed neighbourhood than red edges).

**Observation**

If $b, r \in \mathbb{N}$ such that $b < r < \binom{b+1}{2}$ then a $(b, r)$-flip graph can be constructed.

- Are there any other pairs of values $(b, r)$ such that $(b, r)$-flip graphs exist?

# Characterising 2-flip graphs and sequences

- **Answer:** No.

## Theorem

*Let $b, r \in \mathbb{N}$ such that $b < r$ and let $G$ be an edge coloured graph such that each vertex has $b$ incident blue edges and $r$ incident red edges. If $r \geq \binom{b+1}{2}$ then $G$ is not a $(b, r)$-flip graph.*

- The proof follows from a simple triangle counting argument (realising that there are 6 distinct colourings of a triangle using two colours) and an application of the pigeon-hole principle.
- Observe that as a consequence we have that $b \geq 3$, as for smaller values there does not exist $r \in \mathbb{N}$ such that $b < r < \binom{b+1}{2}$.
- Hence, we have the following characterisation theorem for the case of two colours.

## Theorem

*Let $b, r \in \mathbb{N}$. If $3 \leq b < r < \binom{b+1}{2}$ then there exists a $(b, r)$-flip graph, and both the upper and lower bounds are sharp.*

# Existence of small 2-flip graphs

- Given a $k$-flip sequence $(a_1, \ldots, a_k)$, a problem of interest is that of finding the smallest order $h(a_1, \ldots, a_k)$ of a graph realising it.
- For the case of two colours, our 'simple' construction using the CCP gives a construction on $2r(b+1)$ vertices, giving us that $h(b,r)$ is $O(rb)$. However, we can do much better than this.
- In particular, through *graph packings*, namely of Cayley graphs, we can obtain better bounds. In particular, when $r < b + 2\left\lfloor \frac{b+2}{6} \right\rfloor^2$ then $h(b,r)$ is $\Theta(b+r)$.

**Theorem**

*Let $b, r \in \mathbb{N}$ such that $4 \leq b < r < b + 2\left\lfloor \frac{b+2}{6} \right\rfloor^2$. Then,*

$$h(b,r) \leq 16\left(2 + \left\lfloor \frac{r}{2} \right\rfloor + \left\lfloor \frac{b+2}{2} \right\rfloor - 2\left\lfloor \frac{b+2}{6} \right\rfloor\right)$$

- For $b + 2\left\lfloor \frac{b+2}{6} \right\rfloor^2 \leq r < \binom{b+1}{2}$, $h(b,r)$ is known to be $O\left((b+r)\sqrt{r-b}\right)$. This follows from a slight modification to our original CCP construction.

# Existence of 3-flip graphs

- Whilst knowing necessary and sufficient conditions for the case when $k = 2$, for the case when $k = 3$ we only have the following necessary condition.

**Theorem**

*If $(a_1, a_2, a_3)$ is a 3-flip sequence, then $a_3 < 2(a_1)^2$.*

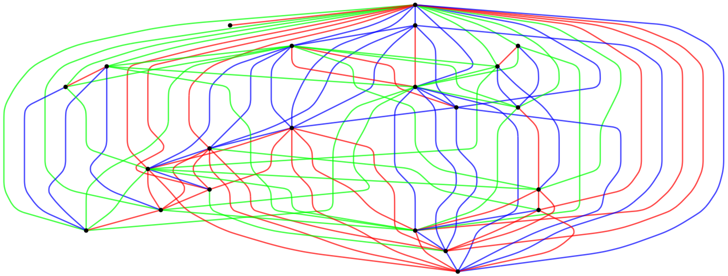

Figure: Illustration of the closed-neighbourhood of a $(6, 7, 8)$-flip graph, namely a Cayley graph on $\mathbb{Z}_{62}$ with connecting sets $B = \{\pm 1, \pm 2, \pm 3\}$, $R = \{\pm 4, \pm 6, \pm 8, 31\}$ and $G = \{\pm 12, \pm 15, \pm 18, \pm 21\}$.

# Constructing $k$-flip graphs: $(r, c)$-constant graphs

- We introduce a new class of graphs called $(r, c)$-constant graphs, which are $r$-regular graphs such that every open neighbourhood has $c$ edges.

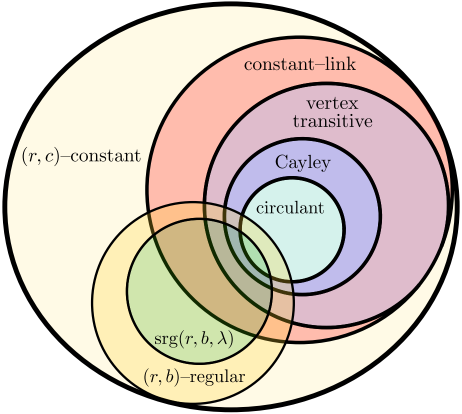


Figure: Interesting sub-families of graphs within the family of $(r, c)$-constant graphs.

# Constructing $k$-flip graphs: $(r, c)$-constant graphs

- Consider a blue coloured $(r_1, c_1)$-constant graph $G$ and a red coloured $(r_2, c_2)$-constant graph $H$ such that $r_1 < r_2$ and $r_1 + c_1 > r_2 + c_2$. By the properties of the CCP, $G \,\square\, H$ is an $(r_1, r_2)$-flip graph.
- This motivates our interest in these graphs. A question of interest is, given some $r$, for what values of $c$ does an $(r, c)$-constant graph. In particular, we have the following result:

**Theorem**

*Let $r \in \mathbb{N}$. For every integer $c$ such that $0 \leq c \leq \frac{r^2}{2} - 5r^{\frac{3}{2}}$, an $(r, c)$-constant graph exists.*

- The proof follows from the work on the feasibility problem of line graphs by Caro, Lauri and Zarb (2023).
- In subsequent work with Caro, a database of such graphs was established.

# Constructing $k$-flip graphs: Flipping intervals

- Given some sufficiently large $b \in \mathbb{N}$, namely $b > 100$, consider the integer interval $[b, b+k]$ where $k = \left\lfloor \frac{1}{4}\left(b^2 - 10b^{\frac{3}{2}}\right)\right\rfloor$. We will show that this is a $k$-flip sequence, using $(r, c)$-constant graphs.
- Let $M_1 = \left\lfloor \frac{b^2}{2} - 5b^{\frac{3}{2}} \right\rfloor$. For $1 \leq j \leq k$, set $H_j$ to be a $(b+j-1, M_1 - 2(j-1))$-constant graph which exists by our result on $(r, c)$-constant graphs.
- Observe that $M_1 \geq 2k \geq 2(j-1)$ for $1 \leq j \leq k$.
- The sequence $b+j-1$ is increasing in $j$, and in particular covers the interval.
- In $H_j$ every closed neighbourhood will have $M_1 + b - j + 1$ edges, which is a decreasing sequence in $j$.
- Therefore, by the properties of the CCP, giving each $H_j$ a unique colour and taking their CCP, we get a $k$-flip graph for the interval $[b, b+k]$.
- Moreover, we go on to show that a flip-graph exists for any subsequence of length $2 \leq s \leq k$ of the interval $[b, b+k]$.

# Existence of $k$-flip graphs

- Recall that for $k \in \{2, 3\}$ we have that the largest colour degree $a_k$ is quadratically bound in the smallest colour degree $a_1$.
- Does this extend to $k \geq 4$? Tantalisingly, using flipping intervals, we have the following sufficient condition...

**Theorem**

*Suppose that $k \geq 2$. Let $3 \leq a_1 < a_2 < \cdots < a_k$ be a sequence of integers such that either $a_k \leq 2a_1 - 2$ or $a_k \leq a_1 + \left\lfloor \frac{1}{4}\left((a_1)^2 - 10(a_1)^{\frac{3}{2}}\right)\right\rfloor$, then $(a_1, \ldots, a_k)$ is a $k$-flip sequence.*

- Here the quadratic bound follows from our previous argument on the construction of flipping intervals and their subsequences.
- **But is it necessary that $a_k$ is quadratically bounded in $a_1$? Intuition suggests that for a flip to occur, $a_k$ must somehow be bounded in $a_1$...**

# Unbounded gaps in $k$-flip sequences

- Rather surprisingly, we have that for $k \geq 4$ constructions exist such that $a_1$ is constant yet $a_k$ can be arbitrarily large.

**Theorem**

*Let $k \in \mathbb{N}, k > 3$. Then there is some constant $m = m(k) \in \mathbb{N}$ such that for all $N \in \mathbb{N}$, there exists a $k$-flip sequence $(a_1, \ldots, a_k)$ such that $a_1 = m$ and $a_k > N$.*

- We are interested in finding the longest possible sub-sequence $(a_1, \ldots, a_{q(k)})$, for some integer $q(k) < k$, such that $a_k$ is independent of this sub-sequence.
- It turns out that these sub-sequences can be fairly long!

**Theorem**

*Let $k \in \mathbb{N}$ such that $k > 3$. Then,*

$$\min\left\{1, \left\lceil \frac{k}{4} \right\rceil - 1\right\} \leq q(k) < \begin{cases} \frac{k}{3} & \text{if } k \mod 3 = 0 \\ \left\lceil \frac{k}{2} \right\rceil & \text{otherwise} \end{cases}$$

# Open problems

- Several problems on flip-graphs are open, some of which are highlighted:

**Problem**

*Determine $h(b, r)$ or at least obtain a nontrivial lower bound.*

**Problem**

*For $k \geq 4$, is there a smallest integer $p(k)$, $\frac{k}{4} \leq p(k) \leq k$, such that $h(a_1, \ldots, a_k)$ is polynomially bound in $a_{p(k)}$?*

**Problem**

*Determine the supremum over all constants $c$ such that there exist infinitely many 3-flip sequences $(a_1, a_2, a_3)$ satisfying $a_3 \geq c(a_1)^2$.*

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