Hamiltonicity of Cayley Graphs and Digraphs

Student: Xandru Mifsud
Supervisor: Prof. Josef Lauri

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L-Università ta' Malta
Faculty of Science
Department
of Mathematics

An algebraic puzzle

Puzzle

Suppose you are given a finite group Γ with n elements and a set $S \subseteq \Gamma$ which generates the group. Construct a sequence $s_1, s_2, \ldots, s_n$ from the elements of S, with repetition allowed, such that every word $s_1 s_2 \ldots s_i$ is a distinct element of Γ.

(Variant)

More so, does there exist some $s_{n+1} \in S$ such that $s_1 = s_1 s_2 \ldots s_n s_{n+1}$?

Example

Consider the dihedral group $D_4 = \langle a, b \mid a^4 = b^2 = 1, ba = a^{-1}b \rangle$ and $S = \{a, b\}$. Then the sequence $s_1 = s_2 = a, s_3 = b, s_4 = s_5 = s_6 = a, s_7 = b, s_8 = a$ is a solution to the puzzle:

$$s_1 = a \qquad s_1 s_2 = a^2 \qquad s_1 s_2 s_3 = a^2 b \qquad s_1 s_2 s_3 s_4 = ab$$

$$s_1 s_2 s_3 s_4 s_5 = b \qquad s_1 s_2 s_3 s_4 s_5 s_6 = a^3 b \qquad s_1 s_2 s_3 s_4 s_5 s_6 s_7 = a^3 \qquad s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 = 1$$

Letting $s_9 = a$ solves the variant of the puzzle, since

$$(s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8) s_9 = (1) a = a = s_1$$

Cayley graphs and digraphs

Cayley digraphs

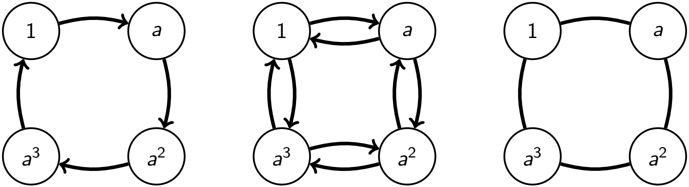
Let Γ be a group and let S be a subset of Γ. The *Cayley digraph* $\overrightarrow{\mathrm{Cay}}(\Gamma; S)$ is a digraph with vertex set Γ, where (g, h) is an arc if, and only if, $\exists s \in S \colon h = gs$.

The set S is often termed as the *connecting set* of the Cayley digraph. For any $s \in S$ and $g \in \Gamma$, we say that the arc (g, gs) in $\overrightarrow{\mathrm{Cay}}(\Gamma; S)$ is *labelled* s.

Let S^{-1} be the set of inverses of the elements in S. We say that S is *inverse–closed* if, and only if, $S^{-1} = S$. When S is inverse–closed, then if (g, h) is an arc in $\overrightarrow{\mathrm{Cay}}(\Gamma; S)$, then (h, g) is also an arc in the Cayley digraph.

Cayley graphs

Let Γ be a group and let $S = S^{-1}$ be a subset of Γ. The *Cayley graph* $\mathrm{Cay}(\Gamma; S)$ is a graph with vertex set Γ, where $\{g, h\}$ is an edge if, and only if, $\exists s \in S \colon h = gs$.



$\overrightarrow{\text{Cay}}\left(Z_4; \{a\}\right)$ $\quad$ $\overrightarrow{\text{Cay}}\left(Z_4; \left\{a, a^{-1}\right\}\right)$ $\quad$ $\text{Cay}\left(Z_4; \left\{a, a^{-1}\right\}\right)$

Figure: Different Cayley (di)graphs associated with the cyclic group Z_4 of order 4.

Association between Cayley graphs and digraphs

Observe that we can consider a Cayley graph as the corresponding Cayley digraph such that every pair of arcs (g, h) and (h, g) is substituted by the edge $\{g, h\}$.

Hence, any result stated for Cayley digraphs applies also to Cayley graphs whenever we consider the connecting set to be inverse–closed.

Properties of Cayley graphs and digraphs

Theorem

Let S be a subset of a group Γ. Then,

1. $\overrightarrow{\text{Cay}}(\Gamma; S)$ *is strongly connected* $\Leftrightarrow$ *the connecting set S generates Γ;*
2. $\overrightarrow{\text{Cay}}(\Gamma; S)$ *is vertex–transitive;*
3. $1 \in S \Leftrightarrow$ *every vertex of* $\overrightarrow{\text{Cay}}(\Gamma; S)$ *has a loop.*

Similarly, (1) – (3) hold for the Cayley graph $\text{Cay}\left(\Gamma; S \cup S^{-1}\right)$.

In the context of (oriented) Hamiltonian paths and cycles, disconnected (di)graphs are of zero interest. More so, loops have no effect on the Hamiltonicity of a (di)graph. Therefore, in light of the above properties, unless otherwise stated we shall assume that the connecting set S of a Cayley (di)graph is a generating set, and that $1 \notin S$.

A different perspective to our puzzle

Recall that the sequence $s_1 = s_2 = a, s_3 = b, s_4 = s_5 = s_6 = a, s_7 = b, s_8 = a$ was a solution to our puzzle for D_4 with generating set $\{a, b\}$. More so, letting $s_9 = a$ solved our variant of the puzzle.

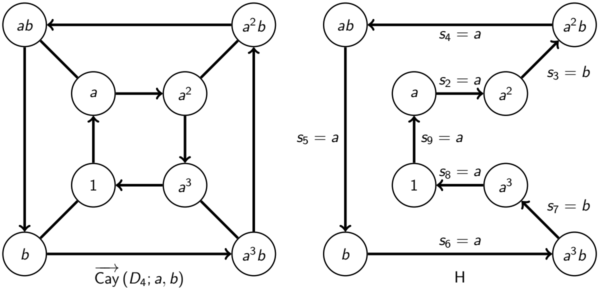


Figure: $\overrightarrow{\text{Cay}}(D_4; \{a, b\})$ and a Hamiltonian cycle H arising from our puzzle's solution, with arcs labelled with the generator in the sequence by which the source vertex travels.

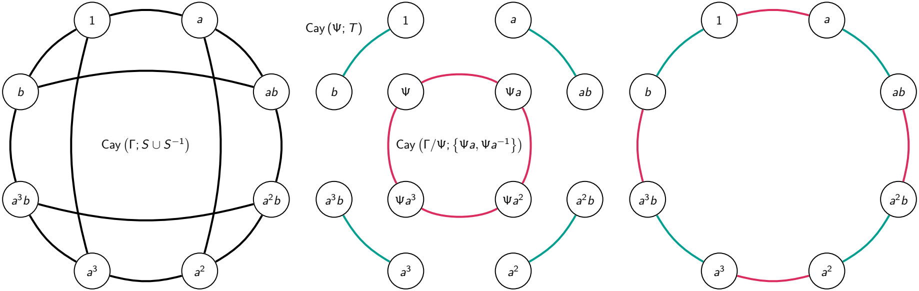
Translation into the language of graph theory

We can pose our puzzle and its variant as follows: Given a group Γ and a generating set S of Γ, does the Cayley digraph $\overrightarrow{\text{Cay}}(\Gamma; S)$ have a Hamiltonian path (cycle)?

The ‘generator extension’ lemma

Consider the Abelian group $\Gamma = \langle a, b | a^4 = b^2 = 1, ab = ba \rangle$ generated by $S = \{a, b\}$[1]. Let $T = \{b\}$; then T generates the subgroup $\Psi = \{1, b\}$ of Γ. Clearly, the quotient group $\Gamma/\Psi = \left\{\Psi, \Psi a, \Psi a^2, \Psi a^{-1}\right\}$ is cyclic and generated by Ψa.

Hence, $\mathrm{Cay}\left(\Gamma/\Psi; \left\{\Psi a, \Psi a^{-1}\right\}\right)$ is trivially a Hamiltonian cycle. Consequently, as shown below, this allows us to join the Hamiltonian paths arising from the Cayley graphs $\mathrm{Cay}\left(\Psi a^i; T\right)$ of each coset of Ψ.



In this manner we construct a Hamiltonian cycle in $\mathrm{Cay}\left(\Gamma; S \cup S^{-1}\right)$.

[1] Note that Γ differs from our previous example D_4, since we allow a and b to commute now.

Marušič (1983) formalised the ideas behind our previous example, by considering for a non–empty proper subset T of a generating set S for an Abelian group, under what conditions does the Hamiltonicity of $\text{Cay}\left(\langle T\rangle\,; T \cup T^{-1}\right)$ extend to the Hamiltonicity of $\text{Cay}\left(\langle S\rangle\,; S \cup S^{-1}\right)$.

We summarise these ideas of Marušič (1983) in the lemma below, which we refer to as the *'generator extension' lemma*.

Theorem ('Generator extension' lemma)

Let Γ be an Abelian group and let S be a generating set of Γ. Let $s \in S$ be of order $m \in \mathbb{N}$, and define $T := S - \{s\}$. Let Ψ be the subgroup of Γ generated by T. If $\text{Cay}\left(\Psi; T \cup T^{-1}\right)$ has a Hamiltonian cycle, then $\text{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.

Problem

Given any Abelian group Γ and a generating set S, when does $\text{Cay}\left(\Gamma; S \cup S^{-1}\right)$ have a Hamiltonian cycle? What about $\overrightarrow{\text{Cay}}\left(\Gamma; S\right)$?

Every Cayley graph on an Abelian group is Hamiltonian

Theorem (Marušič (1983))

For any finite Abelian group Γ *of order at least* 3 *and any generating set* S *of* Γ, $\text{Cay}\left(\Gamma; S \cup S^{-1}\right)$ *has a Hamiltonian cycle.*

Proof.

1. If S has an element a of order ≥ 3, $\exists$ a Hamiltonian cycle in $\text{Cay}\left(\langle a \rangle; \left\{a, a^{-1}\right\}\right)$.
2. If S does not have an element of order at least 3, then since Γ has order ≥ 3, S must contain two elements b and c of order 2. Clearly, $\text{Cay}\left(\langle b, c \rangle; \{b, c\}\right)$ has a Hamiltonian cycle, namely the cycle $(1)(b)(bc)(bcb)(bcbc)$.
3. In either case, extending $\{a\}$ or $\{b, c\}$ respectively by another generator in S, we can apply the 'generator extension' lemma. Repeating inductively until we exhaust all the generators in S, invoking the 'generator extension' lemma for each generator added, the result follows. □

This is arguably one of the most beautiful results in the intersection of graph and group theory, with an equally elegant inductive proof.

A conjecture of Lovász

Lovász's conjecture for Cayley graphs

Every finite connected Cayley graph has a Hamiltonian cycle.

We have just seen that *every* Cayley graph on *any* Abelian group with *any* generating set has a Hamiltonian cycle. This is great evidence that Lovász's conjecture holds in the positive – yet the problem is still wide open for non–Abelian groups!

Throughout the years, stronger evidence has emerged for the non–Abelian case. We note the following two noteworthy results.

Theorem (Witte (1986))

Every connected Cayley digraph on a p–group has an oriented Hamiltonian cycle.

Observe that for inverse–closed generating sets, the result of Witte (1986) implies that every connected Cayley graph on a p–group is Hamiltonian.

Theorem (Pak and Radoičić (2009))

Every finite group Γ of size $|\Gamma| \geq 3$ has a generating set S of size $|S| \leq \log_2 |\Gamma|$, such that $\text{Cay}\left(\Gamma; S \cup S^{-1}\right)$ has a Hamiltonian cycle.

Non–Hamiltonian Cayley digraphs on Abelian groups

Theorem (Rankin (1948))

Let Γ be a finite Abelian group generated by $\{a, b\}$, where $a \neq b$. Let $m = \frac{|\Gamma|}{|\langle ab^{-1}\rangle|}$ and $s \in \mathbb{Z}$ such that $b^m = \left(ab^{-1}\right)^s$. Then $\overrightarrow{\text{Cay}}\left(\Gamma; \{a, b\}\right)$ has an oriented Hamiltonian cycle $\Leftrightarrow \exists k \in \mathbb{Z}$ such that $\gcd\left(k, \circ\left(ab^{-1}\right)\right) = 1$ and $s \leq k \leq s + m$.

Consequently, this gives rise to an infinite–class of Cayley digraphs which are non–Hamiltonian on Abelian groups with a generating set of size 2.

$\overrightarrow{\text{Cay}}\left(\mathbb{Z}_{+ \text{ mod } 12}; \{3, 4\}\right)$ is not Hamiltonian

Consider the additive group modulo 12, $\mathbb{Z}_{+ \text{ mod } 12}$, with generators $a = 3$ and $b = 4$. Then $b^{-1} = 8$ and hence $\left\langle ab^{-1}\right\rangle = \langle 11\rangle = \mathbb{Z}_{+ \text{ mod } 12}$. We show using Rankin's result that $\overrightarrow{\text{Cay}}\left(\mathbb{Z}_{+ \text{ mod } 12}; \{3, 4\}\right)$ does not have an oriented Hamiltonian cycle. Let m, s and k have the same meaning as in the theorem above. Then $m = 1$ and for $s = 8$ we have $b = \left(ab^{-1}\right)^s$. Hence the possible values of k are either 8 or 9. Consequently, $\gcd(8, 12) = 4$ and $\gcd(9, 12) = 3$ *ie.* $\overrightarrow{\text{Cay}}\left(\mathbb{Z}_{+ \text{ mod } 12}; \{3, 4\}\right)$ does not have an oriented Hamiltonian cycle.

Is this the complete picture on the Hamiltonicity of Cayley digraphs for Abelian groups? Certainly not! Firstly, we have the following result and corresponding conjecture – observe how these highlight the importance of the choice of S.

Theorem

Every finite Abelian group has a minimal generating set S, such that $\overrightarrow{\text{Cay}}(\Gamma; S)$ has an oriented Hamiltonian cycle.

Conjecture (Curran and Witte (1985))

For any finite Abelian group Γ and any minimal generating set S of Γ with $|S| \geq 3$, $\overrightarrow{\text{Cay}}(\Gamma; S)$ has an oriented Hamiltonian cycle.

However, we have a much stronger result – if we just demand a Hamiltonian path!

Theorem (Holsztyński and Strube (1978))

For any finite Abelian[a] group Γ and any generating set S of Γ, $\overrightarrow{\text{Cay}}(\Gamma; S)$ has an oriented Hamiltonian path.

[a] The original result is stated in terms of *Dedekind* groups. A group is said to be Dedekind if, and only if, every subgroup is normal. Every Abelian group is Dedekind.

Summary

1. We have posed a classical problem in group theory and translated it to the language of graph theory, namely through Cayley (di)graphs and Hamiltonicity.
2. Introduced a conjecture of Lovász on the Hamiltonicity of Cayley graphs, and presented strong evidence supporting it in the positive.
3. In particular, we introduced one of several techniques used in this area of research: the 'generator extension' lemma.
4. Using this technique, we proved that every Cayley graph on any Abelian group with any generating set has a Hamiltonian cycle.
5. Lastly, we have seen that the same cannot be said for Cayley digraphs of Abelian groups – the best we can do is a Hamiltonian path, but not a Hamiltonian cycle.

Thank you for attending!

A copy of these slides is available at github.com/xmif1/MAT3999

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